The Eigenvalue Shift Technique and Its Eigenstructure Analysis of a Matrix ☆

Chun-Yueh Chiang[1]

Center for General Education, National Formosa University, Huwei 632, Taiwan.

Matthew M. Lin[2,*]

Department of Mathematics, National Chung Cheng University, Chia-Yi 621, Taiwan.

Abstract

The eigenvalue shift technique is the most well-known and fundamental tool for matrix computations. Applications include the search of eigeninformation, the acceleration of numerical algorithms, the study of Google's PageRank. The shift strategy arises from the concept investigated by Brauer [3] for changing the value of an eigenvalue of a matrix to the desired one, while keeping the remaining eigenvalues and the original eigenvectors unchanged. The idea of shifting distinct eigenvalues can easily be generalized by Brauer's idea. However, shifting an eigenvalue with multiple multiplicities is a challenge issue and worthy of our investigation. In this work, we propose a new way for updating an eigenvalue with multiple multiplicities and thoroughly analyze its corresponding Jordan canonical form after the update procedure.

Keywords: eigenvalue shift technique, Jordan canonical form, rank-k updates.

1. Introduction

The eigenvalue shift technique is a will-established method of mathematics in science and engineering. It arises from the research of eigenvalue computations. As we know, the power method is the most common and easiest algorithm to find the dominant eigenvalue for a given matrix $A \in \mathbb{C}^{n \times n}$, but it is impracticable to look for a specific eigenvalue of A. In order to compute one specific eigenvalue, we need to apply the power method to the inverse of the shifted

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matrix $(A - \mu I_n)$ with an appropriately chosen shift value $\mu \in \mathbb{C}$. Here, I_n is an $n \times n$ identity matrix. This is the well-known shifted inverse power method [7]. Early applications of eigenvalue shift techniques are focused on the stability analysis of dynamical systems [1], the computation of the root of a linear or non-linear equation by using preconditioning approaches [13]. Only recently, the study of eigenvalue shift approaches has been widely applied to the study of inverse eigenvalue problems for reconstructing a structured model from pre-scribed spectral data [5, 6, 4, 16], the structure-preserving double algorithm for solving nonsymmetric algebraic matrix Riccati equation [9, 2], and the Google's PageRank problem [11, 17]. Note that the common goal of the applications of the shift techniques stated above is to replace some unwanted eigenvalues so that the stability or acceleration of the prescribed algorithms can be obtained. Hence, a natural idea, the main contribution of our work, is to propose a method for updating the eigenvalue of a given matrix and provide the precise Jordan structures of this updated matrix.

For a matrix $A \in \mathbb{C}^{n \times n}$ and a number μ, if (λ, v) is an eigenpair of A, then matrices $A + \mu I_n$ and μA have the same eigenvector v corresponding to the eigenvalue $\lambda + \mu$ and $\mu\lambda$, respectively. However, these two processes are to translate or scale *all* eigenvalues of A. Our work here is to change a particular eigenvalue, which is enlightened through the following important result given by Brauer [3].

Theorem 1.1. (Brauer). *Let A be a matrix with $Av = \lambda_0 v$ for some nonzero vector v. If r is a vector so that $r^\top v = 1$, then for any scalar λ_1, the eigenvalues of the matrix*

$$\widehat{A} = A + (\lambda_1 - \lambda_0)vr^\top,$$

consist of those of A, except that one eigenvalue λ_0 of A is replaced by λ_1. Moreover, the eigenvector v is unchanged, that is, $\widehat{A}v = \lambda_1 v$.

To demonstrate our motivation, consider an $n \times n$ complex matrix A. Let $A = VJV^{-1}$ be a Jordan matrix decomposition of A, where J is the Jordan normal form of A and V is a matrix consisting of generalized eigenvectors of A. Denote the matrix J as

$$J = \left[\begin{array}{cc} J_k(\lambda) & 0 \\ 0 & J_2 \end{array} \right],$$

where $J_k(\lambda)$ is a $k \times k$ Jordan block corresponding to eigenvalue λ given by

$$J_k(\lambda) := \left[\begin{array}{ccccc} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \lambda & 1 \\ 0 & \cdots & \cdots & 0 & \lambda \end{array} \right] \in \mathbb{C}^{k \times k}$$

and J_2 is an $(n-k) \times (n-k)$ matrix composed of a finite direct sum of Jordan blocks. Then, partition matrices V and V^{-1} conformally as

$$V = \left[\begin{array}{cc} V_1 & V_2 \end{array} \right], (V^{-1})^* = \left[\begin{array}{cc} V_3 & V_4 \end{array} \right],$$

with $V_1, V_3 \in \mathbb{C}^{n \times k}$ and $V_2, V_4 \in \mathbb{C}^{n \times (n-k)}$. A natural idea of updating the eigenvalue appearing at the top left corner of the matrix J, but keeping V and V^{-1} unchanged, is to add a rank-k matrix $\Delta A = V_1 \Delta J_1 V_3^*$ to the original matrix A so that

$$A + \Delta A = V \begin{bmatrix} J_k(\lambda) + \Delta J_1 & 0 \\ 0 & J_2 \end{bmatrix} V^{-1}$$

has the desired eigenvalue. Here, ΔJ_1 is a particular $k \times k$ matrix, which makes $J_k(\lambda) + \Delta J_1$ a new Jordan matrix. Observe that the updated matrix $A + \Delta A$ preserves the structures of matrices V and V^{-1}, but changes the unwanted eigenvalue via the new Jordan matrix $J_k(\lambda) + \Delta J_1$. In other words, if we know the Jordan matrix decomposition in prior, we can update the eigenvalue of A without no difficulty. Note that V_3^* and V_1 are matrices composed of the generalized left and right eigenvectors, respectively, and $V_3^* V_1 = I_k$. In practice, given any generalized left and right eigenvectors corresponding to $J_k(\lambda)$, the condition $V_3^* V_1 = I_k$ is not true in general. Hence, the eigenvalue shift approach stated above cannot not be applied.

In this paper, we want to investigate an eigenvalue shift technique for updating the eigenvalue of the matrix A, provided that partial generalized left and right eigenvectors are given. We show that after the shift technique the first half generalized eigenvectors are kept the same. Indeed, the study of the invariant of the first half generalized eigenvectors is essential for finding the stabilizing solution of an algebraic Riccati Equation and is the so-called *Schur method* or *invariant subspace method* [14, 8]. To advance our research we organize this paper as follows. In Section 2, several useful features of the generalized left and right eigenvectors are discussed. In particular, we investigate the principle of generalized biorthogonality of generalized eigenvectors. This principle is then applied to the study of the eigenvalue shift technique in Section 3 and 4. Finally, the conclusions and some open problems are given in Section 5.

2. The Principle of Generalized Biorthogonality

For a given $n \times n$ square matrix A, let $\sigma(A)$ be the set of all eigenvalues of A. We say that two vectors u and v in $\mathbb{C}^n$ are orthogonal if $u^* v = 0$. In our study, we are seeking the orthogonality of eigenvectors of a given matrix. The feature is know as the *principle of biorthogonality* and is discussed in [10, Theorem 1.4.7] as follows:

Theorem 2.1. *If $A \in \mathbb{C}^{n \times n}$ and if $\lambda_1, \lambda_2 \in \sigma(A)$, with $\lambda_1 \neq \lambda_2$, then any left eigenvector of A corresponding to λ_1 is orthogonal to any right eigenvector of A corresponding to λ_2.*

Theorem 2.1 tells us the principle of biorthogonality with respect to any two distinct eigenvalues. Next, we want to enhance this feature to generalized left and right eigenvectors.

3

Theorem 2.2. *[Generalized Biorthogonality Property] Let $A \in \mathbb{C}^{n \times n}$ and let $\lambda_1, \lambda_2 \in \sigma(A)$. Suppose $\{u_i\}_{i=1}^{p}$ and $\{v_i\}_{i=1}^{q}$ are the generalized left and right eigenvectors corresponding to the Jordan block $J_p(\lambda_1)$ and $J_q(\lambda_2)$, respectively. Then*

(a) *If $\lambda_1 \neq \lambda_2$,*
$$u_i^* v_j = 0, \quad 1 \le i \le p,\, 1 \le j \le q.$$

(b) *If $\lambda_1 = \lambda_2$,*
$$
\begin{aligned}
u_i^* v_j &= u_{i-1}^* v_{j+1}, \quad 2 \le i \le p,\, 1 \le j \le q-1, & (1a)\\
u_i^* v_j &= 0, \quad 2 \le i+j \le \max\{p, q\}. & (1b)
\end{aligned}
$$

PROOF. Let U and V be two matrices defined by
$$U = [\, u_1 \quad \cdots \quad u_p \,] \in \mathbb{C}^{n \times p},\, V = [\, v_1 \quad \cdots \quad v_q \,] \in \mathbb{C}^{n \times q}.$$

By the definitions of $\{u_i\}_{i=1}^{p}$ and $\{v_i\}_{i=1}^{q}$, we have
$$U^* A = J_p^\top(\lambda_1) U^*,\, AV = V J_q(\lambda_2).$$

That is,
$$U^* A V = (U^* V) J_q(\lambda_2) = J_p^\top(\lambda_1)(U^* V). \qquad (2)$$

Define components $x_{i,j} = u_i^* v_j$, $x_{i,0} = 0$, and $x_{0,j} = 0$, for $i = 1, \ldots, p$ and $j = 1, \ldots, q$. Then (2) implies that

$$(\lambda_1 - \lambda_2) x_{i,j} = x_{i-1,j} - x_{i,j-1}, \quad 1 \le i \le p,\, 1 \le j \le q. \qquad (3)$$

It follows from (3) and the assumptions of $x_{i,j}$ that if $\lambda_1 \neq \lambda_2$, then $x_{i,j} = 0$ for $1 \le i \le p$ and $1 \le j \le q$. This proves (a).

By (3), we see that if $\lambda_1 = \lambda_2$, $x_{i-1,j} = x_{i,j-1}$ for $1 \le i \le p$ and $1 \le j \le q$. It follows that $x_{i,j} = 0$, for $2 \le i+j \le \max\{p, q\}$ and elements $x_{i,j}$ coincide on the "matrix diagonals" $i + j = s$ for any $\max\{p, q\} + 1 \le s \le p + q$. This completes the proof of (b). $\qquad \square$

Note that the values $x_{i,j}$ defined in the proof of Theorem 2.2 imply that if $\lambda_1 = \lambda_2$, the matrix $X = [x_{i,j}]_{p \times q}$ is indeed a *lower triangular Hankel matrix*. Also, by (1a), we have the following useful results.

Corollary 2.1. *Let $A \in \mathbb{C}^{n \times n}$ and let $\lambda \in \sigma(A)$. Suppose $\{u_i\}_{i=1}^{p}$ and $\{v_i\}_{i=1}^{q}$ are the generalized left and right eigenvectors corresponding to the Jordan block $J_p(\lambda)$ and $J_q(\lambda)$, respectively. Then, if p and q are even, then*

$$u_i^* v_j = 0, \quad 1 \le i \le \frac{p}{2},\, 1 \le j \le \frac{q}{2}; \qquad (4)$$

if p and q are odd, then

$$u_i^* v_j = 0,\, u_{\frac{p+1}{2}}^* v_j = 0,\, u_i^* v_{\frac{q+1}{2}} = 0, \quad 1 \le i \le \frac{p-1}{2},\, 1 \le j \le \frac{q-1}{2}. \qquad (5)$$

Note that Corollary 2.1 provides only the necessary conditions for two generalized left and right eigenvectors to be orthogonal. It is possible that two generalized eigenvectors are orthogonal, even if they are not fitted in the constraints given in (4) and (5). This phenomenon can be observed by the following two examples. We first provide all possible types of generalized eigenvectors of a Jordan matrix with one and two Jordan blocks, respectively. We then come up with two extreme cases for each Jordan matrix under discussion. One is with the smallest number of orthogonal generalized left and right eigenvectors. The other is with the largest number of orthogonal ones.

Example 2.1. *Let $A = J_{2k}(\lambda)$. Then all of the generalized left and right eigenvectors $\{u_i\}_{i=1}^{2k} \subset \mathbb{C}^{2k}$ and $\{v_i\}_{i=1}^{2k} \subset \mathbb{C}^{2k}$, respectively, can be written as*

$$u_i = \sum_{j=1}^{i} a_{i-j+1} e_{2k-j+1}, \; v_i = \sum_{j=1}^{i} b_{i-j+1} e_j, \; 1 \leq i \leq 2k,$$

where a_i and b_i are arbitrary complex numbers and $a_1 b_1 \neq 0$. Moreover, if $a_i = b_i = 1$ for all i, then

$$u_i^* v_j = 0, \; 1 \leq i, j \leq k,$$
$$u_i^* v_j \neq 0, \; k+1 \leq i, j \leq 2k,$$

and if $a_1 = b_1 = 1$ and $a_i = b_i = 0$ for all $i > 1$, then

$$u_i^* v_j = 0, \; i + j \neq 2k + 1,$$
$$u_i^* v_j \neq 0, \; i + j = 2k + 1.$$

The second example demonstrates the orthogonal properties of the generalized eigenvectors between two Jordan blocks.

Example 2.2. *Let $A = J_k(\lambda) \oplus J_k(\lambda)$. Then all of the generalized left eigenvectors of the first Jordan block $J_k(\lambda)$ (the upper left corner) and the right generalized eigenvectors of the second Jordan block $J_k(\lambda)$ (the lower right corner) can be written as*

$$u_i = \sum_{j=1}^{i} a_j e_{k-j+1} + b_j e_{2k-j+1}, \; v_i = \sum_{j=1}^{i} c_{i-j+1} e_j + d_{i-j+1} e_{k+j}, \; 1 \leq i \leq k,$$

respectively, where a_i, b_i, c_i and d_i are arbitrary complex numbers and $(|a_1|^2 + |b_1|^2)(|c_1|^2 + |d_1|^2) > 0$. Moreover, if $a_i = b_i = c_i = d_i = 1$ for all i, then for all $1 \leq i, j \leq k$

$$u_i^* v_j = 0, \; i + j < k + 1,$$
$$u_i^* v_j \neq 0, \; i + j \geq k + 1,$$

and if $a_i = d_i = 1, b_i = c_i = 0$ for all i, then

$$u_i^* v_j = 0, \; 1 \leq i, j \leq k.$$

Given a matrix $A \in \mathbb{C}^{n \times n}$, let us close this section with the study of the mapping of the resolvent operator $(A - \lambda I_n)^{-1}$ on its generalized left and right eigenvectors. Evidently, the orthogonal property between two generalized left and right eigenvectors will be influenced after the mapping. This influence will play a crucial role in proposing an eigenvalue shift technique later on.

Theorem 2.3. *Let A be a matrix in $\mathbb{C}^{n \times n}$ and let $\lambda_0 \in \sigma(A)$. Suppose $\{u_i\}_{i=1}^{p}$ and $\{v_i\}_{i=1}^{p}$ be the generalized left and right eigenvectors corresponding to $J_p(\lambda_0)$. Let λ be a complex number and $\lambda \notin \sigma(A)$. Then*

(a) For $1 \leq i \leq p$,

$$(A - \lambda I_n)^{-1} v_i = \sum_{j=1}^{i} \frac{(-1)^{i-j} v_j}{(\lambda_0 - \lambda)^{i-j+1}},$$

$$u_i^*(A - \lambda I_n)^{-1} = \sum_{j=1}^{i} \frac{(-1)^{i-j} u_i^*}{(\lambda_0 - \lambda)^{i-j+1}}.$$

(b) For $i + j \leq p$ and $i, j \geq 1$, $u_i^*(A - \lambda I_n)^{-1} v_j = 0$.

PROOF. Set $\lambda \in \mathbb{C}$ and $\lambda \notin \sigma(A)$. Since $(A - \lambda I_n)v_1 = (\lambda_0 - \lambda)v_1$, $(A - \lambda I_n)^{-1}v_1 = \dfrac{v_1}{(\lambda_0 - \lambda)}$. Also, $(A - \lambda I_n)v_2 = (\lambda_0 - \lambda)v_2 + v_1$. It follows that

$$(A - \lambda I_n)^{-1} v_2 = \frac{v_2 - (A - \lambda I_n)^{-1} v_1}{(\lambda_0 - \lambda)} = \sum_{j=1}^{2} \frac{(-1)^{2-j} v_j}{(\lambda_0 - \lambda)^{2-j+1}}.$$

Subsequently, a similar proof can be given without difficulty to different generalized left and right eigenvectors, that is, (a) follows.

Apply Theorem 2.2 and part (a). It is true that $u_i^*(A - \lambda I_n)^{-1} v_j = 0$, for $i + j \leq p$ and $i, j \geq 1$. The result of (b) is established. $\square$

Now we have enough tools to analyze the eigenvalue shift problems. We first establish a result for eigenvalues with even algebraic multiplicities and then generalize it to eigenvalues with odd algebraic multiplicities. We show how to shift a desired eigenvalue of a given matrix without changing the remaining ones.

3. Eigenvalue of Even Algebraic Multiplicities

In this section, we propose a shift technique to move an eigenvalue with even algebraic multiplicities to a desired one. We claim that based on our approach, we can keep parts of generalized eigenvectors unchanged. We also investigate all of the possible Jordan structures after the shift at the end of this section.

Theorem 3.1. *Let $A \in \mathbb{C}^{n \times n}$, let $\lambda_0 \in \sigma(A)$ with algebraic multiplicity $2k$ and geometric multiplicity 1, and let $\{u_i\}_{i=1}^{2k}$ and $\{v_i\}_{i=1}^{2k}$ be the left and right Jordan chains for λ_0. Define two matrices*

$$U := \begin{bmatrix} u_1 & u_2 & \cdots & u_k \end{bmatrix}, \ V := \begin{bmatrix} v_1 & v_2 & \cdots & v_k \end{bmatrix}. \tag{6}$$

If matrices $R^* \in \mathbb{C}^{k \times n}$ and $L \in \mathbb{C}^{n \times k}$ are, respectively, the right and left inverses of V and U^* satisfying

$$U^* L = R^* V = I_k, \tag{7}$$

then the shifted matrix

$$\widehat{A} := A + (\lambda_1 - \lambda_0) R_1 R_2^* \tag{8}$$

where $R_1 := \begin{bmatrix} V & L \end{bmatrix}$ and $R_2 := \begin{bmatrix} R & U \end{bmatrix}$, has the following properties:

(a) *The eigenvalues of $\widehat{A}$ consist of those of A, except that the eigenvalue λ_0 of A is replaced by λ_1.*

(b) *$\widehat{A} v_1 = \lambda_1 v_1$, $u_1^* \widehat{A} = u_1^* \lambda_1$, $\widehat{A} v_{i+1} = \lambda_1 v_{i+1} + v_i$, $u_{i+1}^* \widehat{A} = \lambda_1 u_{i+1}^* + u_i^*$, for $i = 1, \ldots, k-1$. That is,*

$$\widehat{A} V = V J_k(\lambda_1),$$
$$U^* \widehat{A} = J_k^\top(\lambda_1) U^*.$$

PROOF. This proof can be obtained by considering the characteristic polynomial of $\widehat{A}$, that is, if $\lambda \notin \sigma(A)$, then

$$
\begin{aligned}
\det(\widehat{A} - \lambda I_n) &= \det(A - \lambda I_n) \det\left(I_n + (\lambda_1 - \lambda_0) R_1 R_2^* (A - \lambda I_n)^{-1}\right) \qquad (9)\\
&= \det(A - \lambda I_n) \det\left(I_{2k} + (\lambda_1 - \lambda_0) R_2^* (A - \lambda I_n)^{-1} R_1\right) \\
&= \det(A - \lambda I_n) \left(\frac{\lambda_1 - \lambda}{\lambda_0 - \lambda}\right)^{2k} \ (\text{ by Theorem 2.2 and Theorem 2.3 }).
\end{aligned}
$$

Since $\det(A - \lambda I_n)$ is a polynomial with a finite number of zeros, we may choose a small perturbation $\epsilon > 0$ such that $\det(A - (\lambda + \epsilon)I_n) \neq 0$, that is, $A - (\lambda + \epsilon)I_n$ is invertible. This implies that if $\lambda \in \sigma(A)$, we can consider the characteristic polynomial $\det(\widehat{A} - (\lambda + \epsilon)I_n)$ so that

$$\det(\widehat{A} - (\lambda + \epsilon)I_n) = \det(A - (\lambda + \epsilon)I_n) \left(\frac{\lambda_1 - (\lambda + \epsilon)}{\lambda_0 - (\lambda + \epsilon)}\right)^{2k}.$$

Observe that both sides of the above equation are continuous functions of ϵ. By the so-called continuity argument method, letting $\epsilon \to 0$ gives that

$$\det(\widehat{A} - \lambda I_n) = \det(A - \lambda I_n) \left(\frac{\lambda_1 - \lambda}{\lambda_0 - \lambda}\right)^{2k}.$$

It follows that (a) holds.

To prove (b), we see that by Theorem 2.2,

$$\begin{aligned}
\widehat{A}v_1 &= Av_1 + [(\lambda_1 - \lambda_0)R_1 R_2^*]v_1 = \lambda_0 v_1 + (\lambda_1 - \lambda_0)v_1 = \lambda_1 v_1. \\
\widehat{A}v_{i+1} &= Av_i + [(\lambda_1 - \lambda_0)R_1 R_2^*]v_i = \lambda_1 v_{i+1} + v_i, \quad i = 1, \ldots, k-1.
\end{aligned}$$

The same approaches can be carried out to obtain $u_1^* \widehat{A} = u_1^* \lambda_1$ and $u_{i+1}^* \widehat{A} = \lambda_1 u_{i+1}^* + u_i^*$, for $i = 1, \ldots, k-1$. □

Next, it is natural to ask about the eigenstructure of this updated matrix $\widehat{A}$ defined in Theorem 3.1. To this end, we use the notions given in Theorem 3.1. Assume without loss of generality that $A \in \mathbb{C}^{2k \times 2k}$, and that $P = \begin{bmatrix} v_1 & \cdots & v_{2k} \end{bmatrix}$ is a nonsingular matrix so that

$$P^{-1}AP = J_{2k}(\lambda_0),$$

By Theorem 2.2, we know that the matrix product $\begin{bmatrix} U^* \\ U_1^* \end{bmatrix} P \in \mathbb{C}^{2k \times 2k}$ is a lower triangular Hankel matrix, where $U_1 = \begin{bmatrix} u_{k+1} & \cdots & u_{2k} \end{bmatrix}$. This suggests a way for constructing matrices U and V in (6), that is,

$$U = P^{-*}\begin{bmatrix} 0 \\ Q \end{bmatrix} \text{ and } V = P\begin{bmatrix} I_k \\ 0 \end{bmatrix},$$

where Q is some nonsingular and lower triangular Hankel matrix in $\mathbb{C}^{k \times k}$. Also, the right and left inverses of (7) can be denoted by

$$R = P^{-*}\begin{bmatrix} I_k \\ S_1^* \end{bmatrix}, \ L = P\begin{bmatrix} S_2 \\ Q^{-*} \end{bmatrix},$$

where S_1 and S_2 are arbitrary $k \times k$ matrices. It follows from (8) that

$$\begin{aligned}
\widehat{A} &= A + (\lambda_1 - \lambda_0)(VR^* + LU^*) \tag{10} \\
&= P\left(J_{2k}(\lambda_0) + (\lambda_1 - \lambda_0)\left(\begin{bmatrix} I_k & S_1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & S_2 Q^* \\ 0 & I_k \end{bmatrix} \right) \right)P^{-1} \\
&= P\begin{bmatrix} J_k(\lambda_1) & (\lambda_1 - \lambda_0)(S_1 + S_2 Q^*) \\ 0 & J_k(\lambda_1) \end{bmatrix}P^{-1}
\end{aligned}$$

That is,

$$\widehat{A} \sim T := \begin{bmatrix} J_k(\lambda_1) & C \\ 0 & J_k(\lambda_1) \end{bmatrix}, \tag{11}$$

for some matrix $C \in \mathbb{C}^{k \times k}$. Based on (11), the next two results discuss the possible eigenspace and eigenstructure types of the matrix T (i.e., $\widehat{A}$). To facilitate our discussion below, we use $\{e_i\}_{1 \le i \le k}$ to denote the standard basis in $\mathbb{R}^k$ from now on.

8

Lemma 3.1. *Let C be a matrix in $\mathbb{C}^{k \times k}$. If T is a matrix given by*

$$T = \begin{bmatrix} J_k(\lambda) & C \\ 0 & J_k(\lambda) \end{bmatrix}, \tag{12}$$

then T has the eigenspace

$$E_\lambda = \begin{cases} span \left\{ \begin{bmatrix} e_1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ e_1 \end{bmatrix} \right\}, & if\ c_{k,1} = 0,\, Ce_1 = 0. \\[12pt] span \left\{ \begin{bmatrix} e_1 \\ 0 \end{bmatrix}, \begin{bmatrix} N_k Ce_1 \\ e_1 \end{bmatrix} \right\}, & if\ c_{k,1} = 0,\, Ce_1 \neq 0. \\[12pt] span \left\{ \begin{bmatrix} e_1 \\ 0 \end{bmatrix} \right\}, & if\ c_{k,1} \neq 0. \end{cases}$$

corresponding to the eigenvalue λ, where $N_k^\top = \lambda I_k - J_k(\lambda)$ and $C = [c_{i,j}]_{k \times k}$.

PROOF. Given two vectors x_1 and x_2 in $\mathbb{C}^{k \times 1}$, let $v^\top = \begin{bmatrix} x_1^\top & x_2^\top \end{bmatrix}$ be an eigenvector of T with respect to λ. It follows that

$$Tv = \lambda v,$$

or, equivalently,

$$\begin{aligned} (J_k(\lambda) - \lambda I_k)x_1 &= -Cx_2, \\ (J_k(\lambda) - \lambda I_k)x_2 &= 0. \end{aligned} \tag{13}$$

The possible types of eigenspace can be obtained by directly solving (13). $\qquad \blacksquare$

Lemma 3.1 suggests a possible characterization of all Jordan canonical forms of T. Note that if x is an eigenvector of the matrix T associated with a Jordan block $J_p(\lambda)$, then its corresponding Jordan canonical basis γ, *a cycle of generalized eigenvectors of T*, can be expressed as $\gamma = \{v_p, v_{p-1}, \ldots, v_1\}$, where

$$v_i = (T - \lambda I_{2k})^{p-i}(x) \text{ for } i < p \text{ and } v_p = x.$$

In this sense, we classify the possible eigenstructure types by using the notions defined in Lemma 3.1 as follows:

Case 1. $c_{k,1} = 0, Ce_1 = 0$. Then, $T \sim J_k(\lambda) \oplus J_k(\lambda)$ with two cycles of generalized eigenvectors taken as

$$\begin{aligned} \gamma_1 &= \left\{ \begin{bmatrix} e_1 \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} e_k \\ 0 \end{bmatrix} \right\}, \\[12pt] \gamma_2 &= \left\{ \begin{bmatrix} 0 \\ e_1 \end{bmatrix}, \begin{bmatrix} N_k Ce_2 \\ e_2 \end{bmatrix}, \ldots, \begin{bmatrix} \sum_{i=2}^{k} N_k^{k-i+1} Ce_i \\ e_k \end{bmatrix} \right\}. \end{aligned}$$

9

Case 2. $c_{k,1} = 0, Ce_1 \neq 0$. Then $T \sim J_k(\lambda) \oplus J_k(\lambda)$ with two cycles of generalized eigenvectors taken as

$$
\gamma_1 = \left\{ \begin{bmatrix} e_1 \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} e_k \\ 0 \end{bmatrix} \right\},
$$

$$
\gamma_2 = \left\{ \begin{bmatrix} N_k Ce_1 \\ e_1 \end{bmatrix}, \ldots, \begin{bmatrix} \sum_{i=1}^{k} N_k^{k-i+1} Ce_i \\ e_k \end{bmatrix} \right\}.
$$

Case 3. $c_{k,1} \neq 0$. Then $T \sim J_{2k}(\lambda)$ with the cycle of generalized eigenvectors taken as

$$
\gamma = \left\{ \begin{bmatrix} c_{k,1} e_1 \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} c_{k,1} e_k \\ 0 \end{bmatrix}, \right.
$$

$$
\left. \begin{bmatrix} N_k Ce_1 \\ e_1 \end{bmatrix}, \ldots, \begin{bmatrix} \sum_{i=1}^{k} N_k^{k-i+1} Ce_i \\ e_k \end{bmatrix} \right\}.
$$

Here, we have identified all possible eigenstructure types of the matrix T. This observation leads directly to the following result.

Theorem 3.2. *Let $\widehat{A}$ be the matrix defined by (10). Then, the Jordan canonical form of the matrix $\widehat{A}$ is either $J_k(\lambda) \oplus J_k(\lambda)$ or $J_{2k}(\lambda)$.*

We shall now give an example to demonstrate the result of Theorem 3.1 and Theorem 3.2.

Example 3.1. *Let $A = J_4(1) \oplus J_2(3) \in \mathbb{R}^{6 \times 6}$ and $U = \begin{bmatrix} e_4 & e_3 \end{bmatrix}$ and $V = \begin{bmatrix} e_1 & e_2 \end{bmatrix}$ be the first half of the generalized left and right eigenvectors of A corresponding to the eigenvalue $\lambda = 1$. We then want to change the eigenvalue $\lambda = 1$ to $\lambda = 2$ and observe the subsequent Jordan canonical forms.*

(a) *From (7), if we take the right and left inverses R and L to be*

$$
R = V = \begin{bmatrix} e_1 & e_2 \end{bmatrix}, L = U = \begin{bmatrix} e_4 & e_3 \end{bmatrix},
$$

then the shifted matrix $\widehat{A}$ satisfies

$$
\widehat{A} := A + (2-1) R_1 R_2^* = J_4(2) \oplus J_2(3),
$$

where $R_1 = \begin{bmatrix} V & L \end{bmatrix}$ and $R_2 = \begin{bmatrix} R & U \end{bmatrix}$.

(b) *From (7), if we take the right and left inverses R and L to be*

$$
R = \begin{bmatrix} e_1 & e_2 - e_3 \end{bmatrix}, L = U = \begin{bmatrix} e_4 & e_3 \end{bmatrix},
$$

then the shifted matrix $\widehat{A}$ satisfies

$$
\widehat{A} := A + (2-1) R_1 R_2^* = J_2(2) \oplus J_2(2) \oplus J_2(3),
$$

where $R_1 = \begin{bmatrix} V & L \end{bmatrix}$ and $R_2 = \begin{bmatrix} R & U \end{bmatrix}$.

Note that Theorem 3.2 also implies that the geometric multiplicity of $\widehat{A}$, defined in (10), is at most two. On the other hand, an analogous approach can seemingly be used to derive a shift technique and characterize the corresponding eigenstructure for the eigenvalue with odd algebraic multiplicities. Indeed, this analysis for the odd case is much more complicated due to the orthogonal property caused by the middle eigenvector as it can be seen in Theorem 2.2 and a different approach is needed for our discussion.

4. Eigenvalue with Odd Algebraic Multiplicities

Let us now consider the eigenvalue with odd algebraic multiplicities. If $p = q$ and p, q are odd integers, observe two full rank matrices $U = \begin{bmatrix} u_1 & \cdots & u_p \end{bmatrix}$ and $V = \begin{bmatrix} v_1 & \cdots & v_p \end{bmatrix}$. It follows from (1a) that the diagonal elements of the lower triangular Hankel matrix U^*V is constant. Thus, the inner product $u^*_{\frac{p+1}{2}} v_{\frac{p+1}{2}} \neq 0$. With this in mind, the following result shows how to change an eigenvalue with odd algebraic multiplicities.

Theorem 4.1. *Let $A \in \mathbb{C}^{n \times n}$, and λ_0 be an eigenvalue of A having algebraic multiplicity $2k + 1$ and geometric multiplicity 1, and let $\{u_i\}_{i=1}^{2k+1}$ and $\{v_i\}_{i=1}^{2k+1}$ be the left and right Jordan chains for λ_0. Define two matrices*

$$U := \begin{bmatrix} u_1 & u_2 & \cdots & u_k \end{bmatrix}, V := \begin{bmatrix} v_1 & v_2 & \cdots & v_k \end{bmatrix} \tag{14}$$

and a vector $r = \dfrac{u_{k+1}}{v^*_{k+1} u_{k+1}}$. If matrices $R^* \in \mathbb{C}^{k \times n}$ and $L \in \mathbb{C}^{n \times k}$ are, respectively, the right and left inverses of V and U^* satisfying

$$R^*V = U^*L = I_k, \tag{15}$$

then the shifted matrix

$$\widehat{A} := A + (\lambda_1 - \lambda_0)R_1 R_2^* \tag{16}$$

where $R_1 := \begin{bmatrix} V & v_{k+1} & L \end{bmatrix}$ and $R_2 := \begin{bmatrix} R & r & U \end{bmatrix}$, has the following properties:

(a) *The eigenvalues of $\widehat{A}$ consist of those of A, except that the eigenvalue λ_0 of A is replaced by λ_1.*

(b) *$\widehat{A}v_1 = \lambda_1 v_1$, $u_1^*\widehat{A} = u_1^*\lambda_1$, $\widehat{A}v_{i+1} = \lambda_1 v_{i+1} + v_i$, $u_{i+1}^*\widehat{A} = \lambda_1 u_{i+1}^* + u_i^*$, for $i = 1, \ldots, k - 1$. That is to say,*

$$\widehat{A}V = V J_k(\lambda_1)$$
$$U^*\widehat{A} = J_k^\top(\lambda_1)U^*$$

11

PROOF. Since

$$R_2^*(A - \lambda I)^{-1}R_1 = \begin{bmatrix} R^*(A - \lambda I)^{-1}V & R^*(A - \lambda I_n)^{-1}v_{k+1} & R^*(A - \lambda I)^{-1}L \\ 0 & \dfrac{1}{\lambda_0 - \lambda} & r^*(A - \lambda I)^{-1}L \\ 0 & 0 & U^*(A - \lambda I)^{-1}L \end{bmatrix},$$

it follows from Theorem 2.2 and Theorem 2.3 that the characteristic polynomial of $\widehat{A}$ satisfies

$$\begin{aligned}
\det(\widehat{A} - \lambda I_n) &= \det(A - \lambda I_n)\det\left(I_n + (\lambda_1 - \lambda_0)R_1 R_2^*(A - \lambda I_n)^{-1}\right) \\
&= \det(A - \lambda I_n)\det\left(I_{2k+1} + (\lambda_1 - \lambda_0)R_2^*(A - \lambda I_n)^{-1}R_1\right) \\
&= \det(A - \lambda I_n)\left(\frac{\lambda_1 - \lambda}{\lambda_0 - \lambda}\right)^{2k+1},
\end{aligned}$$

if $\lambda \notin \sigma(A)$. For the case $\lambda \in \sigma(A)$, a small perturbation of λ is applied to make $A - \lambda I$ nonsingular. Thus, a similar proof like the one in Theorem 3.1 is followed. This proves (a).

We see by direct computation and Theorem 2.2 that

$$\widehat{A}v_1 = Av_1 + [(\lambda_1 - \lambda_0)R_1 R_2^*]v_1 = \lambda_0 v_1 + (\lambda_1 - \lambda_0)v_1 = \lambda_1 v_1,$$

$$\widehat{A}v_{i+1} = \lambda_0 v_{i+1} + v_i + (\lambda_1 - \lambda_0)R_1\begin{bmatrix} e_{i+1} \\ 0 \end{bmatrix} = \lambda_1 v_{i+1} + v_i, \quad i = 1,\ldots,k-1.$$

In an analogous way, we can obtain $u_1^*\widehat{A} = u_1^*\lambda_1$ and $u_{i+1}^*\widehat{A} = \lambda_1 u_{i+1}^* + u_i^*$, for $i = 1,\ldots,k-1$, and (b) follows. $\qquad\square$

Theorem 4.1 provides us a way to update the eigenvalue of a given Jordan block of odd size. Consequently, we are interested in analyzing possible eigenstructure types of the shifted matrix $\widehat{A}$. We start this discussion in an analogous way from Section 3 by using the notions defined in Theorem 4.1. Assume for simplicity that the given matrix A is of size $(2k+1) \times (2k+1)$ and $P = \begin{bmatrix} v_1 & \cdots & v_{2k+1} \end{bmatrix}$ is a nonsingular matrix such that

$$P^{-1}AP = J_{2k+1}(\lambda_0).$$

It follows that $V = P\begin{bmatrix} I_k \\ 0 \end{bmatrix}$. Set $U = P^{-*}\begin{bmatrix} 0 \\ Q \end{bmatrix}$ for some lower triangular Hankel matrix $Q \in \mathbb{C}^{k \times k}$, and the vector $v_{k+1} = P\hat{e}_{k+1}$, where $\hat{e}_{k+1}$ is a unit vector in $\mathbb{R}^{2k+1}$ with a 1 in position $k+1$ and 0's elsewhere. Corresponding to formula (15) and the vector r given in Theorem 4.1, we define

$$R = P^{-*}\begin{bmatrix} I_k \\ S_1^* \end{bmatrix}, \quad L = P\begin{bmatrix} S_2 \\ Q^{-*} \end{bmatrix}, \quad r = P^{-*}\begin{bmatrix} 0 \\ w^*/\overline{w_1} \end{bmatrix},$$

where S_1^* and S_2 are arbitrary matrices in $\mathbb{C}^{(k+1) \times k}$, $w = [w_i]_{k \times 1}$ is a vector in $\mathbb{C}^{k+1}$, and $\overline{w_1}$ is the complex conjugate of w_1. Then the shifted matrix (16)

satisfies

$$
\widehat{A} = A + (\lambda_1 - \lambda_0)(VR^* + v_{k+1}r^* + LU^*) \tag{17}
$$

$$
= P\left[J_{2k+1}(\lambda_0) + (\lambda_1 - \lambda_0)\left(\begin{bmatrix} I_k & S_1 \\ 0 & 0 \end{bmatrix} + \hat{e}_{k+1}\begin{bmatrix} 0 & w/w_1 \end{bmatrix} + \begin{bmatrix} 0 & S_2Q^* \\ 0 & I_k \end{bmatrix}\right)\right]P^{-1}
$$

$$
= P\begin{bmatrix} J_k(\lambda_1) & (\lambda_1-\lambda_0)\hat{s}_1 & (\lambda_1-\lambda_0)(S_1\begin{bmatrix} 0 \\ I_k \end{bmatrix} + \begin{bmatrix} I_k & 0 \end{bmatrix}S_2Q^*) \\ 0 & \lambda_1 & (\lambda_1-\lambda_0)(\hat{s}_2 + \begin{bmatrix} w_2 & \dots & w_{k+1} \end{bmatrix}) \\ 0 & 0 & J_k(\lambda_1) \end{bmatrix}P^{-1},
$$

where $\hat{s}_1$ is the first column of S_1 and $\hat{s}_2$ is the last row of S_2. This implies that after the shift approach the matrix $\widehat{A}$ is similar to an upper triangular matrix

$$
S := \begin{bmatrix} J_k(\lambda) & a & C \\ 0 & \lambda & b^\top \\ 0 & 0 & J_k(\lambda) \end{bmatrix}, \tag{18}
$$

for some matrix $C \in \mathbb{C}^{k \times k}$ and vectors $a, b \in \mathbb{C}^k$. This above similarity property allows us to discuss the eigeninformation of the shifted matrix $\widehat{A}$. That is, the eigeninformation of $\widehat{A}$ can be studied indirectly by considering the upper triangular matrix T of (18)

Lemma 4.1. *Let C be a matrix in $\mathbb{C}^{k \times k}$ and let a, b be two vectors in $\mathbb{C}^k$. If S is given by*

$$
S = \begin{bmatrix} J_k(\lambda) & a & C \\ 0 & \lambda & b^\top \\ 0 & 0 & J_k(\lambda) \end{bmatrix},
$$

13

then S has the eigenspace

$$
E_\lambda = \begin{cases}
span\left\{ \begin{bmatrix} e_1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} N_k C e_1 \\ 0 \\ e_1 \end{bmatrix} \right\}, & \text{if } b_1 = 0, a_k \neq 0, c_{k,1} = 0. \\[3em]
span\left\{ \begin{bmatrix} e_1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} N_k(\frac{-c_{k,1}}{a_k} a + C e_1) \\ \frac{-c_{k,1}}{a_k} \\ e_1 \end{bmatrix} \right\}, \\[1em]
\quad \text{if } b_1 = 0, a_k \neq 0, c_{k,1} \neq 0. \\[2em]
span\left\{ \begin{bmatrix} e_1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} N_k a \\ 1 \\ 0 \end{bmatrix} \right\}, \\[1em]
\quad \text{if } b_1 = 0, a_k = 0, c_{k,1} \neq 0 \text{ or } b_1 \neq 0, a_k = 0. \\[2em]
span\left\{ \begin{bmatrix} e_1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} N_k(a + C e_1) \\ 1 \\ e_1 \end{bmatrix}, \begin{bmatrix} N_k C e_1 \\ 0 \\ e_1 \end{bmatrix} \right\}, \\[1em]
\quad \text{if } b_1 = 0, a_k = 0, c_{k,1} = 0. \\[2em]
span\left\{ \begin{bmatrix} e_1 \\ 0 \\ 0 \end{bmatrix} \right\}, & \text{if } b_1 \neq 0, a_k \neq 0.
\end{cases}
\tag{19}
$$

corresponding to the eigenvalue λ, where $N_k^\top = \lambda I_k - J_k(\lambda)$, $b_1 = e_1^\top b$, $a_k = e_k^\top a$, and $C = [c_{i,j}]_{k \times k}$.

PROOF. Corresponding to the eigenvalue λ, let $v^\top = \begin{bmatrix} x_1^\top & x_2^\top & x_3^\top \end{bmatrix}$ be an eigenvector of S, where $x_1, x_3 \in \mathbb{C}^k$ and $x_2 \in \mathbb{C}$. It is true that the eigenvector v satisfies

$$Sv = \lambda v,$$

or, equivalently,

$$
\begin{aligned}
(J_k(\lambda) - \lambda I_k)x_1 &= -x_2 a - C x_3, \\
0 x_2 &= -b^\top x_3, \\
(J_k(\lambda) - \lambda I_k)x_3 &= 0.
\end{aligned}
$$

We then have the all possible types of eigenspace by discussing the cases given in (19) step by step. $\qquad \square$

Similarly, we proceed to discuss all possible Jordan canonical forms of this shifted matrix (16) through the discussion of this special block upper triangular matrix S given in (18). One point should be made clear first. Due to the effects of vectors a and b in S. It is too complicate to evaluate the generalized

eigenvectors in an analogous way as we did in Section 3. We, therefore, seek to determine the Jordan canonical forms by means of some kind of matrix transformation so that the structure and eigeninfomration of S are simplified and unchanged respectively.

With this in mind, we start by choosing an invertible matrix

$$Y = \begin{bmatrix} I_k & y & W \\ 0 & 1 & z^\top \\ 0 & 0 & I_k \end{bmatrix} \in \mathbb{C}^{(2k+1)\times(2k+1)}.$$

It is easy to check that

$$Y^{-1} = \begin{bmatrix} I_k & -y & -W + yz^\top \\ 0 & 1 & -z^\top \\ 0 & 0 & I_k \end{bmatrix}.$$

We then transform matrix S in (18) by using Y and Y^{-1} such that

$$YSY^{-1} = \begin{bmatrix} J_k(\lambda) & a - (J_k(\lambda) - \lambda I)y & \widetilde{C} \\ 0 & \lambda & b^\top + z^\top(J_k(\lambda) - \lambda I) \\ 0 & 0 & J_k(\lambda) \end{bmatrix},$$

where

$$\widetilde{C} = [\tilde{c}_{i,j}]_{k\times k} = WJ_k(\lambda) - J_k(\lambda)W + C + yb^\top - az^\top + (J_k(\lambda) - \lambda I)yz^\top.$$

Specify the vectors $a = [a_i]_{k\times 1}$ and $b = [b_i]_{k\times 1}$ by their components, and let the matrix $D = [d_{i,j}]_{k\times k} = C + yb^\top - az^\top + (J_k(\lambda) - \lambda I)yz^\top$.

Notice, first, that if we choose

$$y^\top = \begin{bmatrix} y_1 & a_1 & a_2 & \dots & a_{k-1} \end{bmatrix} \text{ and } z^\top = \begin{bmatrix} -b_2 & -b_3 & \dots & -b_k & z_k \end{bmatrix},$$

then for any y_1 and $z_k \in \mathbb{C}$ we have

$$\tilde{a} := a - (J_k(\lambda) - \lambda I)y = \begin{bmatrix} 0 \\ a_k \end{bmatrix} \text{ and } \tilde{b}^\top := b^\top + z^\top(J_k(\lambda) - \lambda I) = \begin{bmatrix} b_1 & 0 \end{bmatrix}. \quad (20)$$

Next, observe that

$$(WJ_k(\lambda) - J_k(\lambda)W)\, e_i = \begin{cases} \begin{bmatrix} -w_{2,1} \\ \vdots \\ -w_{k,1} \\ 0 \end{bmatrix}, & i = 1, \\[2em] \begin{bmatrix} w_{1,i-1} - w_{2,i} \\ \vdots \\ w_{k-1,i-1} - w_{k,i} \\ w_{k,i-1} \end{bmatrix}, & 2 \leq i \leq k, \end{cases}$$

15

where $W = [w_{i,j}]_{k \times k}$. We then want to eliminate the elements in $\widetilde{C}$ by choosing

$$w_{i+1,1} = d_{i,1}, \quad 1 \le i \le k-1,$$
$$w_{i+1,j} = w_{i,j-1} + d_{i,j}, \quad 1 \le i \le k-1, \, 2 \le j \le k,$$

so that

$$\tilde{c}_{i,j} \;=\; 0, \quad 1 \le i < k, \, 1 \le j \le k, \tag{21a}$$

$$\tilde{c}_{k,j} \;=\; \sum_{\ell=0}^{j-1} d_{k-\ell,j-\ell}, \quad 1 \le j \le k. \tag{21b}$$

From formulae (20) and (21), it follows that by choosing appropriate vectors y, z and a matrix W, we can simplify the matrix S such that

$$S \sim \widetilde{S} := \begin{bmatrix} J_k(\lambda) & \tilde{a} & \widetilde{C} \\ 0 & \lambda & \tilde{b}^{\top} \\ 0 & 0 & J_k(\lambda) \end{bmatrix},$$

where $\tilde{a} = a_k e_k$, $\tilde{b} = b_1 e_1$ are vectors in $\mathbb{C}^k$, and $\widetilde{C} = [\tilde{c}_{i,j}]_{k \times k}$ satisfies $\tilde{c}_{i,j} = 0$, for $1 \le i \le k-1$ and $1 \le j \le k$. Note that $\widetilde{C}$ is a matrix with all entries equal to zero except the ones on the last row and is said to be in *lower concentrated form* [12]. This observation can be used to classify all possible eigenstructure types of S though the discussion of $\widetilde{S}$. We list all the classifications as follows:
Case 1. $b_1 \neq 0, a_k \neq 0$.

1a. If $\tilde{c}_{k,1} = \ldots = \tilde{c}_{k,i} = 0$ and $\tilde{c}_{k,i+1} \neq 0$, for $0 \le i \le k-1$, then $S \sim J_{2k+1}(\lambda)$ with the cycle of generalized eigenvectors taken as

$$\gamma \;=\; \left\{ \begin{bmatrix} b_1 a_k e_1 \\ 0 \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} b_1 a_k e_k \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ b_1 \\ 0 \end{bmatrix}, \right.$$
$$\left. \begin{bmatrix} 0 \\ 0 \\ e_1 \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ 0 \\ e_i \end{bmatrix}, \begin{bmatrix} 0 \\ \tau_1 \\ f_1 \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ \tau_{k-i} \\ f_{k-i} \end{bmatrix} \right\},$$

where

$$f_1 \;=\; e_{i+1} \in \mathbb{C}^k, \quad f_j = e_{i+j} + \frac{1}{b_1} \sum_{s=1}^{j-1} \tau_s e_{j-s} \in \mathbb{C}^k, \quad 2 \le j \le k-i,$$

$$\tau_j \;=\; \frac{-e_k^{\top} \widetilde{C} f_j}{a_k} \in \mathbb{C}, \quad 1 \le j \le k-i.$$

1b. If $\tilde{c}_{k,1} = \ldots = \tilde{c}_{k,k} = 0$, then $S \sim J_{2k+1}(\lambda)$ with the cycle of generalized eigenvectors taken as

16

$$\gamma \;=\; \left\{ \begin{bmatrix} b_1 a_k e_1 \\ 0 \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} b_1 a_k e_k \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ b_1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ e_1 \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ 0 \\ e_k \end{bmatrix} \right\}.$$

Case 2. $b_1 = 0, a_k \neq 0$.

2a. If $\tilde{c}_{k,1} = \ldots = \tilde{c}_{k,k-1} = 0$ and $\tilde{c}_{k,k} \neq 0$, then $S \sim J_{k+1}(\lambda) \oplus J_k(\lambda)$ with the cycles of generalized eigenvectors taken as

$$\gamma_1 \;=\; \left\{ \begin{bmatrix} a_k e_1 \\ 0 \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} a_k e_k \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\},$$

$$\gamma_2 \;=\; \left\{ \begin{bmatrix} 0 \\ 0 \\ e_1 \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ 0 \\ e_{k-1} \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{-\tilde{c}_{k,k}}{a_k} \\ e_k \end{bmatrix} \right\}.$$

2b. If $\tilde{c}_{k,1} = \ldots = \tilde{c}_{k,i} = 0$ and $\tilde{c}_{k,i+1} \neq 0$, for $0 \leq i < k - 1$, then $S \sim J_{2k-i}(\lambda) \oplus J_{i+1}(\lambda)$ with the cycles of generalized eigenvectors taken as

$$\gamma_1 \;=\; \left\{ \begin{bmatrix} m_1 \\ n_1 \end{bmatrix}, \ldots, \begin{bmatrix} m_{2k-i} \\ n_{2k-i} \end{bmatrix} \right\},$$

$$\gamma_2 \;=\; \left\{ \begin{bmatrix} 0 \\ 0 \\ e_1 \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ 0 \\ e_i \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{-\tilde{c}_{k,i+1}}{a_k} \\ e_{i+1} \end{bmatrix} \right\}.$$

where

$$m_j \;=\; \begin{cases} \tilde{c}_{k,i+1} e_j, & 1 \leq j \leq k, \\ 0_{k \times 1}, & k+1 \leq j \leq 2k - i. \end{cases}$$

$$n_j \;=\; \begin{cases} 0_{(k+1) \times 1}, & 1 \leq j < k - i + 1, \\ \left[0 \quad \sum\limits_{s=0}^{j-k+i-1} \alpha_s e_{j-k+i-s}^\top \right]^\top, & k - i + 1 \leq j < 2k - 2i - 1, \\ \left[0 \quad \sum\limits_{s=0}^{k-i-2} \alpha_s e_{j-k+i-s}^\top \right]^\top, & 2k - 2i - 1 \leq j \leq 2k - i - 1, \\ \left[\frac{-1}{a_k} \sum\limits_{s=0}^{k-i-2} \alpha_s \tilde{c}_{k,k-s} \quad \sum\limits_{s=0}^{k-i-2} \alpha_s e_{k-s}^\top \right]^\top, & j = 2k - i, \end{cases}$$

$$\alpha_0 \;=\; 1, \quad \alpha_j = \frac{-1}{\tilde{c}_{k,i+1}} \sum\limits_{s=0}^{j-1} \alpha_s \tilde{c}_{k,i+2+s} \in \mathbb{C}, \quad 1 \leq j \leq k - i - 1.$$

Here, if $k = i + 2$, we should ignore the second case define in n_j by using the third one directly.

2c. If $\tilde{c}_{k,1} = \ldots = \tilde{c}_{k,k} = 0$, then $S \sim J_{k+1}(\lambda) \oplus J_k(\lambda)$ with the cycles of generalized eigenvectors taken as

$$\gamma_1 = \left\{ \begin{bmatrix} a_k e_1 \\ 0 \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} a_k e_k \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\},$$

$$\gamma_2 = \left\{ \begin{bmatrix} 0 \\ 0 \\ e_1 \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ 0 \\ e_k \end{bmatrix} \right\}.$$

Case 3.$b_1 \neq 0, a_k = 0$.

3a. If $\tilde{c}_{k,1} = \ldots = \tilde{c}_{k,i} = 0$ and $\tilde{c}_{k,i+1} \neq 0$, for $0 \leq i \leq k-1$, then $S \sim J_{2k-i}(\lambda) \oplus J_{i+1}(\lambda)$ with the cycles of generalized eigenvectors taken as

$$\gamma_1 = \left\{ \begin{bmatrix} m_1 \\ n_1 \end{bmatrix}, \ldots, \begin{bmatrix} m_{2k-i} \\ n_{2k-i} \end{bmatrix} \right\},$$

$$\gamma_2 = \left\{ \begin{bmatrix} 0 \\ b_1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ e_1 \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ 0 \\ e_i \end{bmatrix} \right\}.$$

where

$$m_j = \begin{cases} \tilde{c}_{k,i+1} e_j, & 1 \leq j \leq k, \\ 0_{k \times 1}, & k+1 \leq j \leq 2k-i. \end{cases}$$

$$n_j = \begin{cases} 0_{(k+1) \times 1}, & 1 \leq j < k-i, \\ \begin{bmatrix} b_1 & 0 \end{bmatrix}^\top, & j = k-i, \\ \begin{bmatrix} b_1 \alpha_{j-k+i} & \sum\limits_{s=0}^{j-k+i-1} \alpha_s e_{j-k+i-s}^\top \end{bmatrix}^\top, & k-i+1 \leq j < 2k-2i, \\ \begin{bmatrix} 0 & \sum\limits_{s=0}^{k-i-1} \alpha_s e_{j-k+i-s} \end{bmatrix}^\top, & 2k-2i \leq j \leq 2k-i, \end{cases}$$

$$\alpha_0 = 1, \quad \alpha_j = \frac{-1}{\tilde{c}_{k,i+1}} \sum_{s=0}^{j-1} \alpha_s \tilde{c}_{k,i+2+s} \in \mathbb{C}, \quad 1 \leq j \leq k-i-1.$$

Here, if $k = i+1$, we should ignore the first case and replace the third case define in n_j by using the forth one directly.

3b. If $\tilde{c}_{k,1} = \ldots = \tilde{c}_{k,k} = 0$, then $S \sim J_{k+1}(\lambda) \oplus J_k(\lambda)$ with the cycles of generalized eigenvectors taken as

$$\gamma_1 = \left\{ \begin{bmatrix} 0 \\ b_1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ e_1 \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ 0 \\ e_k \end{bmatrix} \right\},$$

$$\gamma_2 = \left\{ \begin{bmatrix} e_1 \\ 0 \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} e_k \\ 0 \\ 0 \end{bmatrix} \right\}.$$

18

Case 4. $\tilde{b}_1 = 0, \tilde{a}_k = 0$.

4a. If $\tilde{c}_{k,1} \neq 0$, then $S \sim J_{2k}(\lambda) \oplus J_1(\lambda)$ with the cycles of generalized eigen-vectors taken as

$$\gamma_1 = \left\{ \begin{bmatrix} \tilde{c}_{k,1}e_1 \\ 0 \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} \tilde{c}_{k,1}e_k \\ 0 \\ 0 \end{bmatrix}, \right.$$

$$\left. \begin{bmatrix} 0 \\ 0 \\ \psi_1 e_1 \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ 0 \\ \sum\limits_{s=1}^{k} \psi_s e_{k-s+1} \end{bmatrix} \right\},$$

$$\gamma_2 = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

where

$$\psi_1 = 1, \quad \psi_j = \frac{-1}{\tilde{c}_{k,1}} \sum_{s=1}^{j-1} \psi_s \tilde{c}_{k,j-s+1} \in \mathbb{C}, \text{ for } j = 2, \ldots, k.$$

4b. If $\tilde{c}_{k,1} = \ldots = \tilde{c}_{k,i} = 0$ and $\tilde{c}_{k,i+1} \neq 0$, for $1 \leq i \leq k-1$, then $S \sim J_{2k-i}(\lambda) \oplus J_i(\lambda) \oplus J_1(\lambda)$, with the cycles of generalized eigenvectors taken as

$$\gamma_1 = \left\{ \begin{bmatrix} m_1 \\ 0 \\ n_1 \end{bmatrix}, \ldots, \begin{bmatrix} m_{2k-i} \\ 0 \\ n_{2k-i} \end{bmatrix} \right\},$$

$$\gamma_2 = \left\{ \begin{bmatrix} 0 \\ 0 \\ e_1 \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ 0 \\ e_i \end{bmatrix} \right\},$$

$$\gamma_3 = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\},$$

where

$$m_j = \begin{cases} \tilde{c}_{k,i+1}e_j, & 1 \leq j \leq k, \\ 0_{k \times 1}, & k+1 \leq j \leq 2k-i. \end{cases}$$

$$n_j = \begin{cases} 0_{(k+1) \times 1}, & 1 \leq j < k-i+1, \\ \sum\limits_{s=0}^{j-k+i-1} \alpha_s e_{j-k+i-s}, & k-i+1 \leq j < 2k-2i, \\ \sum\limits_{s=0}^{k-i-1} \alpha_s e_{j-k+i-s}, & 2k-2i \leq j \leq 2k-i, \end{cases}$$

$$\alpha_0 = 1, \quad \alpha_j = \frac{-1}{\tilde{c}_{k,i+1}} \sum_{s=0}^{j-1} \alpha_s \tilde{c}_{k,i+2+s} \in \mathbb{C}, \text{ for } j = 1, \ldots, k-i-1.$$

19

Here, if $k = i + 1$, we should ignore the second case define in n_j by using the third one directly.

4c. If $\tilde{c}_{k,1} = \ldots = \tilde{c}_{k,k} = 0$, then $A \sim J_k(\lambda) \oplus J_k(\lambda) \oplus J_1(\lambda)$, with the cycles of generalized eigenvectors taken as

$$\gamma_1 = \left\{ \begin{bmatrix} 0 \\ 0 \\ e_1 \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ 0 \\ e_k \end{bmatrix} \right\},$$

$$\gamma_2 = \left\{ \begin{bmatrix} e_1 \\ 0 \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} e_k \\ 0 \\ 0 \end{bmatrix} \right\},$$

$$\gamma_3 = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

Theorem 4.2. *Let $\widehat{A}$ be the matrix defined by* (17). *Then, the Jordan canonical form of the matrix $\widehat{A}$ is one of the types: $J_{2k+1}(\lambda)$, $J_{2k-i+1}(\lambda) \oplus J_i(\lambda)$, and $J_{2k-i}(\lambda) \oplus J_i(\lambda) \oplus J_1(\lambda)$, for $1 \leq i \leq k$.*

Now, we have all possible eigenstructure types of the matrix S. In view of such classifications above, it turns out that the geometric multiplicity of $\widehat{A}$, defined in (16), is at most three.

5. Conclusions and Open Problems

The methods developed in this paper are used to shift an eigenvalue with algebraic multiplicities greater than 1, in the sense that the first half of corresponding generalized eigenvectors are kept unchanged. From the point of view of applications, the approach has the advantage that one can apply this shift technique to speed up or stabilizing a given numerical algorithm.

It is true that there are many different kinds of eigenvalue shift problems for a wide range of applications in science and engineering. For example, in our recent work [15], we apply the shift approach to remove two zero eigenvalues embedded in a nonsymmetric algebraic Riccati equation. After the shift, the speed of convergence of the simple iteration algorithm for finding the minimal nonnegative solution is significantly improved. Indeed, this application is a special case corresponding to Theorem 3.1 with $k = 1$.

That is, in this work, we propose a more general way to shift eigenvalues of a given matrix with multiple algebraic multiplicities. But, it is worthy of note that the computation of the Jordan form is extremely sensitive to perturbations. It appears to be an interesting open and challenge problem to propose a reliably numerical method to compute the Jordan form of a given matrix with floating-point arithmetic. On the other hand, in most cases we are required to shift partial (not all) eigenvalues of a given Jordan block, change multiple eigenvalues simultaneously, or replace complex conjugate eigenvalues of a real matrix. All these questions are under investigation and will be reported elsewhere.

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